PRINCIPLES OF ANALYSIS
CONDENSED LECTURE NOTES

PAUL L. BAILEY

1. Natural Numbers

We assume intuitive familiarity with the natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$.

Assertion 1. (Peano Axioms)
The natural numbers *are a set* $\mathbb{N}$ *satisfying the following properties.*

(N1) $1 \in \mathbb{N}$
(N2) $n \in \mathbb{N} \Rightarrow n^+ \in \mathbb{N}$
(N3) $1 \neq n^+$ *for any* $n \in \mathbb{N}$
(N4) $m^+ = n^+ \Rightarrow m = n$
(N5) *If* $A \subset \mathbb{N}$ *such that* $1 \in A$ *and* $n \in A \Rightarrow n^+ \in A$*, then* $A = \mathbb{N}$

We call these properties the Peano Axioms.

Proposition 1. (Principal of Mathematical Induction)
Let p_n *be a proposition, for each* $n \in \mathbb{N}$*. Suppose*

(I1) p_1;
(I2) $p_n \Rightarrow p_{n+1}$.

Then p_n *is true for every* $n \in \mathbb{N}$*.*

Definition 1. Let $a \in \mathbb{C}$. We say that a is *algebraic* if there exists a polynomial $f(x)$ with coefficients in $\mathbb{Z}$ such that $f(a) = 0$.

Proposition 2. (Rational Zeros Theorem)
Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$*, with* $a_i \in \mathbb{Z}$ *for all* i *and* $a_n \neq 0$*. Let* $\frac{m}{n} \in \mathbb{Q}$*, with* $m, n \in \mathbb{Z}$*,* $n > 0$*, and* $\gcd(m, n) = 1$*. If* $f(\frac{m}{n}) = 0$*, then* m *divides* a_0 *and* n *divides* m*.*

Problem 1. Show that $3^n \leq n^3$ for all $n \in \mathbb{N}$.

Problem 2. Show that

$$\sec\left(\frac{\pi}{8}\right) = \sqrt{4 - 2\sqrt{2}}$$

is an algebraic number.

Date: December 9, 2005.

2. Ordered Fields

2.1. Fields.

Definition 2. A *field* is a set F together with binary operators

$$+ : F \times F \to F \quad \text{and} \quad \cdot : F \times F \to F$$

satisfying:

(A1) $a + (b + c) = (a + b) + c$ for all $a, b, c \in F$
(A2) $a + b = b + a$ for all $a, b \in F$
(A3) $\exists 0 \in F$ such that $a + 0 = a$ for all $a \in F$
(A4) $\forall a \in F \, \exists - a \in F$ such that $a + (-a) = 0$
(M1) $a(bc) = (ab)c$ for all $a, b, c \in F$
(M2) $ab = ba$ for all $a, b \in F$
(M3) $\exists 1 \in F$ such that $a \cdot 1 = a$ for all $a \in F$
(M4) $\forall a \in F \setminus \{0\} \, \exists a^{-1} \in F$ such that $aa^{-1} = 1$
(DL) $a(b + c) = ab + bc$ for all $a, b, c \in F$

2.2. Ordered Fields.

Definition 3. An *ordered field* is a field F together with a relation

$$\leq \subset F \times F$$

satisfying:

(O1) $a \leq b$ or $b \leq a$ for all $a, b \in F$
(O2) $a \leq b$ and $b \leq a$ implies $a = b$ for all $a, b \in F$
(O3) $a \leq b$ and $b \leq c$ implies $a \leq c$ for all $a, b, c \in F$
(O4) $a \leq b$ implies $a + b \leq b + c$ for all $a, b, c \in F$
(O5) $a \leq b$ and $0 \leq c$ implies $ac \leq bc$ for all $a, b, c \in F$

Remark 1. Let F be an ordered field and let $x, y \in F$. Then

- $x < y$ means $x \leq y$ and $x \neq y$;
- $x \geq y$ means $y \leq x$;
- $x > y$ means $y < x$;
- $x < y < z$ means $x < y$ and $y < z$.

Remark 2. Let F be an ordered field; then F contains 0 and 1. Since F is ordered, the set of elements obtained by adding 1 to itself is infinite, and since F is closed under addition, F contains $\mathbb{N}$. Since F is closed under additive inverses, F contains $\mathbb{Z}$. Since F is closed under multiplicative inverses, F contains $\mathbb{Q}$.

2.3. Complete Ordered Fields.

Definition 4. Let F be an ordered field. Let $S \subset F$ and let $b \in F$.

We say that b is an *upper bound* for S if $s \leq b$ for every $s \in S$.

We say that b is a *lower bound* for S if $b \leq s$ for every $s \in S$.

We say that b is the *least upper bound (supremum)* of S, and write $b = \sup S$, if

(1) $s \leq b$ for every $s \in S$;
(2) if $s \leq c$ for every $s \in S$, then $b \leq c$.

We say that b is the *greatest lower bound (infimum)* of S, and write $b = \inf S$, if

(1) $b \leq s$ for every $s \in S$;
(2) if $c \leq s$ for every $s \in S$, then $c \leq b$.

Definition 5. (Completeness Axiom)
Let F be an ordered field. We say that F is *complete* if

(CA) every subset of F which is bounded above has a least upper bound.

Proposition 3. *Let F be a complete ordered field. Then every subset of F which is bounded below has a greatest lower bound.*

Proposition 4. (Archimedean Property)
Let F be a complete ordered field. Let $a, b \in F$ with $0 < a < b$. Then there exists $n \in \mathbb{N}$ such that $na < b$.

Proposition 5. (Density of $\mathbb{Q}$)
Let F be a complete ordered field. Let $a, b \in F$ with $a < b$. Then there exists $q \in \mathbb{Q}$ such that $a < q < b$.

Assertion 2. *The* real numbers *are a set $\mathbb{R}$ whose algebraic and order structure produce a complete ordered field.*

2.4. **Problems.**

Problem 3. Let A and B be bounded sets of real numbers with $B \subset A$.

(a) Show that $\sup B \leq \sup A$.
(b) Show that $\inf B \geq \inf A$.

Problem 4. Let A and B be bounded sets of real numbers. Define

$$A + B = \{x \in \mathbb{R} \mid x = a + b \text{ for some } a \in A, b \in B\}.$$

(a) Show that $\sup(A + B) = \sup A + \sup B$.
(b) Show that $\inf(A + B) = \inf A + \inf B$.

Problem 5. Let A and B be bounded sets of real numbers. Define

$$A - B = \{x \in \mathbb{R} \mid x = a - b \text{ for some } a \in A, b \in B\}.$$

(a) Show that $\sup(A - B) = \sup A - \inf B$.
(b) Show that $\inf(A - B) = \inf A - \sup B$.

Problem 6. Let A and B be bounded sets of positive real numbers. Define

$$A * B = \{x \in \mathbb{R} \mid x = ab \text{ for some } a \in A, b \in B\}.$$

(a) Show that $\sup(A * B) = \sup A \sup B$.
(b) Show that $\inf(A * B) = \inf A \inf B$.

Solution Part **(a)**. Let $ab \in A * B$. Then $a \leq \sup A$ and $b \leq \sup B$. Since a and b are nonnegative, $ab \leq (\sup A)(\sup B)$, which shows that $(\sup A)(\sup B)$ is an upper bound for the set $A * B$. Thus $\sup A * B \leq (\sup S)(\sup T)$.

Suppose $\sup A * B < (\sup A)(\sup B)$. Then $\sup A * B / \sup B < \sup A$. Select $a \in A$ such that $\sup A * B / \sup B < a \leq \sup A$. Then $\sup A * B / a < \sup B$. Select $b \in B$ such that $\sup A * B / a < b \leq \sup B$. Then $\sup A * B < ab$, a contradiction. □

Problem 7. Let A and B be bounded sets of positive real numbers. Define

$$A/B = \{x \in \mathbb{R} \mid x = \frac{a}{b} \text{ for some } a \in A, b \in B\}.$$

(a) Show that $\sup(A/B) = \frac{\sup A}{\inf B}$.
(b) Show that $\inf(A/B) = \frac{\inf A}{\sup B}$.

Problem 8. Let $\alpha \in \mathbb{R}$ and let $A = \{r \in \mathbb{Q} \mid r < \alpha\}$. Show that $\sup A = \alpha$.

3. Sequences

3.1. **Sequences.**

Definition 6. Let A be a set. A *sequence* in A is a function $a : \mathbb{N} \to A$. We write a_n to mean $a(n)$, and we write $(a_n)_{n=1}^{\infty}$, or simply (a_n), to denote the function a.

We are primarily interested in sequences of real numbers, i.e., sequences in $\mathbb{R}$.

Definition 7. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers and let $p \in \mathbb{R}$. We say that $(a_n)_{n=1}^{\infty}$ *converges to* p

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \ni \; n \geq N \; \Rightarrow \; |a_n - p| < \epsilon.$$

In this case, we say that p is a *limit point* of $(a_n)_{n=1}^{\infty}$.

Proposition 6. *Let* $(a_n)_{n=1}^{\infty}$ *be a sequence in* $\mathbb{R}$ *and let* $p_1, p_2 \in \mathbb{R}$.
If $(a_n)_{n=1}^{\infty}$ *converges to* p_1 *and to* p_2*, then* $p_1 = p_2$.

Proof. Suppose not, and set $d = |p_1 - p_2|$; then d is positive. Let $\epsilon = \frac{d}{4}$. Then by definition of limit, there exist positive integers N_1 and N_2 such that $n \geq N_1$ implies that $|a_n - p_1| < \epsilon$, and $n \geq N_2$ implies that $|a_n - p_2| < \epsilon$.

Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$,

$$\begin{aligned}
d &= |p_1 - p_2| \\
&= |p_1 - a_n + a_n - p_2| \\
&= |p_1 - a_n| + |a_n - p_2| \quad \text{by the Triangle Inequality} \\
&= |a_n - p_1| + |a_n - p_2| \\
&\leq \epsilon + \epsilon \\
&= \frac{d}{2}.
\end{aligned}$$

This is a contradiction; thus $p_1 = p_2$. □

Thus limits are unique when they exist, justifying the article *the* limit instead of "a limit point". We write $p = \lim_{n \to \infty} a_n$, or simply $p = \lim a_n$, or even $a_n \to p$ to denote the fact that $(a_n)_{n=1}^{\infty}$ converges to p. If a sequence has a limit, we say that it is *convergent*; otherwise it is *divergent*.

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. The *image* of $(a_n)_{n=1}^{\infty}$ is the image of the sequence as a function, that is, it is the set

$$\{a_n \mid n \in \mathbb{N}\}.$$

Note that there is much more information in a sequence than in its image; for example, the sequences $(1+(-1)^n)_{n=1}^{\infty}$ and $(0, 2, 0, 0, 2, 0, 0, 0, 2, 0, 0, 0, 0, 2, \dots)$ have the same image; the common image is $\{0, 2\}$, a set containing two elements.

3.2. **Arithmetic of Sequences.**

Proposition 7. *Let $(a_n)_{n=1}^{\infty}$ be a convergent sequence in $\mathbb{R}$, and let $k \in \mathbb{R}$. Then the sequence $(ka_n)_{n=1}^{\infty}$ converges, and*

$$\lim_{n\to\infty} ka_n = k \lim_{n\to\infty} a_n.$$

Proof. Let $\epsilon > 0$, and let $p = \lim_{n\to\infty} a_n$. Since $a_n \to p$, there exists $N \in \mathbb{N}$ such that

$$|a_n - p| < \frac{\epsilon}{k}.$$

Then

$$|ka_n - kp| < \epsilon.$$

□

Proposition 8. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be convergent sequences of real numbers. Then the sequence $(a_n + b_n)_{n=1}^{\infty}$ converges, and*

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

Proposition 9. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be convergent sequences of real numbers. Then the sequence $(a_n b_n)_{n=1}^{\infty}$ converges, and*

$$\lim_{n\to\infty} (a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n).$$

Proposition 10. *Let $(a_n)_{n=1}^{\infty}$ be a convergent sequence of nonzero real numbers whose limit is not zero. Then the sequence $(\frac{1}{a_n})_{n=1}^{\infty}$ converges, and*

$$\frac{1}{\lim_{n\to\infty} a_n} = \lim_{n\to\infty} \left(\frac{1}{a_n}\right).$$

3.3. **Bounded Sequences.**

Definition 8. Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. We say that (a_n) is *bounded above* if there exists $a \in \mathbb{R}$ such that $a \geq s_n$ for every $n \in \mathbb{N}$. We say that (a_n) is *bounded below* if there exists $b \in \mathbb{R}$ such that $b \leq a_n$ for every $n \in \mathbb{N}$. We say that $(a_n)_{n=1}^{\infty}$ is *bounded* if it is bounded above and bounded below.

Equivalently, $(a_n)_{n=1}^{\infty}$ is bounded if there exists $b > 0$ such that $a_n \in [-b, b]$ for every $n \in \mathbb{N}$.

Proposition 11. *Every convergent sequence in $\mathbb{R}$ is bounded.*

Proof. Let $(a_n)_{n=1}^{\infty}$ be a convergent sequence with limit p. Let N be so large that for $n \geq N$ we have $|a_n - p| < 1$. And $|p|$ to both sides of this inequality and apply the triangle inequality to get, for every $n \geq N$,

$$|a_n| \leq |a_n - p| + |p| < 1 + |p|.$$

There are only finitely many terms of the sequence between a_1 and a_{N-1}; set

$$M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |p|\}.$$

Then $M \geq a_n$ for every $n \in \mathbb{N}$, so $(a_n)_{n=1}^{\infty}$ is bounded. □

Proposition 12. *Let* $(s_n)_{n=1}^{\infty}$ *be a sequence in* $\mathbb{R}$ *which converges to* p*, and let* $a, b \in \mathbb{R}$ *with* $a < b$*.*

(a) *If* $s_n \geq a$ *for every* $n \in \mathbb{N}$*, then* $p \geq a$*.*
(b) *If* $s_n \leq b$ *for every* $n \in \mathbb{N}$*, then* $p \leq b$*.*
(c) *If* $s_n \in [a, b]$ *for every* $n \in \mathbb{N}$*, then* $p \in [a, b]$*.*

Proof. In this proof, we use the fact that if $x \leq y + \epsilon$ for every $\epsilon > 0$, then $x \leq y$. To see this, suppose that $x > y$, and let $\epsilon = \frac{x-y}{2}$; then $y + \epsilon = x - \epsilon$, so $x > y + \epsilon$.

Suppose that $s_n \geq a$ for every $n \in \mathbb{N}$. To show that $a \leq p$, it suffices to show that $a \leq p+\epsilon$ for every $\epsilon > 0$. Thus let $\epsilon > 0$; since (s_n) converges to p, there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |s_n - p| < \epsilon$. Thus $-\epsilon < s_n - p < \epsilon$, so $s_n < p + \epsilon$. Since $a \leq s_n$, transitivity of order implies that $a < p + \epsilon$. Since this is true for every $\epsilon > 0$, we have $a \leq p$.

That $p \leq b$ can be proved similarly.

Finally, if $s_n \in [a, b]$, we have $a \leq s_n \leq b$ for every $n \in \mathbb{N}$. Combining parts **(a)** and **(b)** tells us that $a \leq p \leq b$, which is equivalent to $p \in [a, b]$. □

Proposition 13. *Let* $(a_n)_{n=1}^{\infty}$ *and* $(b_n)_{n=1}^{\infty}$ *be sequences in* $\mathbb{R}$ *such that* $a_n \leq b_n$ *for every* $n \in \mathbb{N}$*. If they both converge, then* $\lim a_n \leq \lim b_n$*.*

Proof. Let $a = \lim a_n$ and $b = \lim b_n$; suppose by way of contradiction that $b < a$. Set $\epsilon = \frac{b-a}{2}$; then there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $|a_n - a| < \epsilon/2$, and there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $|b_n - b| < \epsilon/2$. Let $N = \max\{N_1, N_2\}$; then by an application of the triangle inequality, $b_n < a_n$, a contradiction. □

Proposition 14. (Squeeze Law)
Let (a_n)*,* (b_n)*, and* (s_n) *be sequences in* $\mathbb{R}$ *such that* $a_n \leq s_n \leq b_n$ *for all* $n \in \mathbb{N}$*. If* $\lim a_n = \lim b_n = p$*, then* (s_n) *converges to* p*.*

Proof. Let $\epsilon > 0$. Note that for any $n \in \mathbb{N}$, since $a_n \leq s_n \leq b_n$ we have

$$|s_n - a_n| = s_n - a_n \leq b_n - a_n = |b_n - a_n|.$$

Since $\lim a_n = p$, there exists $N_1 \in \mathbb{N}$ such that $|a_n - p| < \frac{\epsilon}{3}$ for $n \geq N_1$.
Since $\lim b_n = s$, there exists $N_2 \in \mathbb{N}$ such that $|b_n - p| < \frac{\epsilon}{3}$ for $n \geq N_2$.
Let $N = \max\{N_1, N_2\}$. Now for $n \geq N$, we have

$$|b_n - a_n| = |b_n - p + p - a_n| \leq |b_n - p| + |a_n - p| < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}.$$

Then for $n \geq N$, we have

$$|s_n - p| = |s_n - a_n + a_n - p| \leq |s_n - a_n| + |a_n - p| \leq |b_n - a_n| + |a_n - p| < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This shows that $\lim s_n = p$. □

3.4. **Monotone Sequences.**

Definition 9. Let $(s_n)_{n=1}^{\infty}$ be a sequence of real numbers.
We say that (s_n) is *increasing* if

$$m \leq n \Rightarrow s_m \leq s_n.$$

We say that (s_n) is *decreasing* if

$$m \leq n \Rightarrow a_m \geq a_n.$$

We say that (s_n) is *monotone* if it is either increasing or decreasing.

Note that to check if the sequence (s_n) is increasing, if suffices to check that $s_{n+1} \geq s_n$ for every $n \in \mathbb{N}$. In this case, the definition above will follow by induction. The analogous comment holds for the condition of decreasing.

Theorem 1. (Monotone Convergence Principle)
Every bounded monotone sequence of real numbers converges.

Proof. Suppose that $(s_n)_{n=1}^{\infty}$ is bounded. Also assume that it is increasing; the proof for decreasing will be analogous. Let $S = \{s_n \mid n \in \mathbb{N}\}$ be the image of the sequence, and set $u = \sup S$. Since S is bounded, $u \in \mathbb{R}$. Clearly $s_n \leq u$ for every $n \in \mathbb{N}$. We show that $\lim s_n = u$.

Let $\epsilon > 0$. Since $u - \epsilon$ is not an upper bound for S, there exists $s \in S$ such that $u - \epsilon < s \leq u$. Now $s = s_N$ for some $N \in \mathbb{N}$, and since $(s_n)_{n=1}^{\infty}$ is increasing, we have $u - \epsilon < s_n < u$ for every $n \geq N$. Thus $|s_n - u| < \epsilon$ for $n \geq N$; this shows that (s_n) converges to u. □

3.5. **Limits Superior and Inferior.**

Proposition 15. *Let $(s_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. Set*

$$u_N = \sup\{s_n \mid n \geq N\} \quad \textit{and} \quad v_N = \inf\{s_n \mid n \geq N\}.$$

Then $(u_n)_{n=1}^{\infty}$ is a bounded decreasing sequence and $(v_n)_{n=1}^{\infty}$ is a bounded increasing sequence. Each of these sequences converges.

Proof. Since (s_n) is a bounded sequence, the sets $\{s_n \mid n \geq N\}$ are bounded sets, so u_N and v_N exist as real numbers for all $N \in \mathbb{N}$, and in fact if $S = \{s_n \mid n \in \mathbb{N}\}$, then $\inf S \leq v_N \leq u_N \leq \sup S$ for every $N \in \mathbb{N}$. Thus the sequences (u_N) and (v_N) are bounded sequences.

To show that these sequences are monotone, we use the general fact that if $A, B \subset \mathbb{R}$ and $B \subset A$, then $\sup B \leq \sup A$ and $\inf B \geq \inf A$.

In our case, select $N \in \mathbb{N}$ and let $A = \{s_n \mid n \geq N\}$ and $B = \{s_n \mid n \geq N+1\}$. Then $B \subset A$, so $\sup B \leq \sup A$, which is to say, $u_{N+1} \leq u_N$. Thus (u_N) is a decreasing sequence. Similarly, (v_N) is an increasing sequence.

Thus (u_N) and (v_N) are bounded monotone sequences, and so are convergent by the Monotone Convergence Principal. □

Definition 10. Let $(s_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. Define the *limit superior* of (s_n) to be

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n \mid n \geq N\}$$

and the *limit inferior* of (s_n) to be

$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n \mid n \geq N\}.$$

Proposition 16. *Let $(s_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers.*
Then $\liminf s_n \leq \limsup s_n$.

Proof. For every $N \in \mathbb{N}$, we have $\inf\{s_n \mid n \geq N\} \leq \sup\{s_n \mid n \geq N\}$. The result follows from Proposition 13. □

Proposition 17. *Let* $(s_n)_{n=1}^{\infty}$ *be a sequence of real numbers.*

(a) *If* (s_n) *converges to* s*, then* $\liminf s_n = s = \limsup s_n$*.*
(b) *If* $\liminf s_n = \limsup s_n$*, then* (s_n) *converges.*

Proof. We again use the fact that if $x \leq y + \epsilon$ for every $\epsilon > 0$, then $x \leq y$.

Suppose that $(s_n)_{n=1}^{\infty}$ converges to a real number s. Let $\epsilon > 0$. We wish to show that $\limsup s_n \leq s + \epsilon$ for every $\epsilon > 0$, whence $\limsup s_n \leq s$.

Since $s_n \to s$, there exists $N \in \mathbb{N}$ such that $|s_n - s| < \epsilon$ for $n \geq N$. It follows that $\sup\{s_n \mid n \geq N\} < s + \epsilon$. Since $(\sup\{s_n \mid n \geq N\})_{N=1}^{\infty}$ is a decreasing sequence, we have $\limsup s_n < s + \epsilon$. Therefore $\limsup s_n \leq s$.

Similarly, $s \leq \liminf s_n$, so

$$s \leq \liminf s_n \leq \limsup s_n \leq s,$$

so

$$\liminf s_n = s = \limsup s_n.$$

Now suppose that $\liminf s_n = \limsup s_n$, and label this common value s. We want to show that $\lim s_n = s$.

Let $\epsilon > 0$. Since $s = \limsup s_n$, there exists $N_1 \in \mathbb{N}$ such that

$$|\sup\{s_n \mid n \geq N_1\} - s| < \epsilon.$$

In particular, $\sup\{s_n \mid n \geq N_1\} < s + \epsilon$, so $s_n < s + \epsilon$ for $n \geq N_1$. Similarly, since $s = \liminf s_n$, there exists $N_2 \in \mathbb{N}$ such that $s_n > s - \epsilon$ for $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$, we have $s - \epsilon < s_n < s + \epsilon$, that is, $|s_n - s| < \epsilon$. Thus $s_n \to s$. $\square$

3.6. **Cauchy Sequences.**

Definition 11. Let $(s_n)_{n=1}^{\infty}$ be a sequence of real numbers. We say that $(s_n)_{n=1}^{\infty}$ is a *Cauchy sequence* if

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \ni \; m, n \geq N \Rightarrow |s_m - s_n| < \epsilon.$$

Proposition 18. *Let* $(s_n)_{n=1}^{\infty}$ *be a Cauchy sequence. Then* $(s_n)_{n=1}^{\infty}$ *is bounded.*

Proof. Since $(s_n)_{n=1}^{\infty}$ is Cauchy, there exists $N \in \mathbb{N}$ such that if $m, n \geq N$, then $|s_m - s_n| < 1$. In particular, for every $n \geq N$, we have $|s_n - s_N| < 1$. Set

$$M = \max\{s_1, s_2, \ldots, s_{N-1}, s_N + 1\}.$$

Then $s_n \in [-M, M]$ for every $n \in \mathbb{N}$. $\square$

Theorem 2. (Cauchy Convergence Criterion)
A sequence of real numbers converges if and only if it is a Cauchy sequence.

Proof. We prove each direction of the double implication.

($\Rightarrow$) Assume that the sequence (s_n) is convergent. Let $\epsilon > 0$, and set $s = \lim s_n$. Then there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|s_n - s| < \epsilon/2$. Then for $m, n \geq N$, we have

$$\begin{aligned}|s_m - s_n| &= |s_m - s + s - s_n| \\ &= |s_m - s| + |s_n - s| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

($\Leftarrow$) Assume that the sequence (s_n) is a Cauchy sequence. Then it is bounded, and so its limit superior and inferior exist as real numbers. By a previous proposition, it suffices to show that $\liminf s_n = \limsup s_n$.

Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that if $m, n \geq N$, then $|s_m - s_n| < \epsilon$. In particular, $|s_n - s_N| < \frac{\epsilon}{2}$ for all $n \geq N$, so $s_N + \frac{\epsilon}{2}$ is an upper bound for $\{s_n \mid n \geq N\}$. Thus $\sup\{s_n \mid n \geq N\} \leq s_N + \frac{\epsilon}{2}$, and therefore $\limsup s_n \leq s_N + \frac{\epsilon}{2}$. Similarly $\liminf s_n \geq s_N - \frac{\epsilon}{2}$. Rearranging these inequalities gives

$$\limsup s_n - \frac{\epsilon}{2} \leq s_N \leq \liminf s_n + \frac{\epsilon}{2},$$

or

$$0 \leq \limsup s_n - \liminf s_m < \epsilon.$$

Since ϵ is arbitrary, we have $\limsup s_n = \liminf s_n$. □

3.7. **Problems.**

Problem 9. Let

$$a_n = \frac{5n^2 + 1}{2n^2 - 3}.$$

Let $\epsilon > 0$. Find $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow |a_n - \frac{5}{2}| < \epsilon.$$

Problem 10. Let $b, c \in \mathbb{R}$ with $b \geq 1$ and $c \geq 0$. Set $d = \frac{b+\sqrt{b^2+4c}}{2}$. Let $x_n = 1$ and $x_{n+1} = \sqrt{bx + c}$.

(a) Use induction to show that $1 \leq x_n \leq d$.
(b) Use induction to show that $x_n \leq x_{n+1}$.
(c) Show that (x_n) converges to d.

Problem 11. Let $(a_n)_{n=1}^{\infty}$ be a convergent sequence of real numbers, and let $A = \{a_n \mid n \in \mathbb{N}\}$. Show that $\lim_{n\to\infty} a_n \leq \sup A$.

Problem 12. Let $(a_n)_{n=1}^{\infty}$ be a sequence in $[a, b]$, where $a, b \in \mathbb{R}$ and $a < b$. Show that if (a_n) converges to p, then $p \in [a, b]$.

Problem 13. Let $(s_n)_{n=1}^{\infty}$ be a sequence of nonzero real numbers such that $\lim_{n\to\infty} |s_n|$ converges to a positive real number. Show that there exists $m > 0$ such that $|s_n| > m$ for all n. (This is a Lemma for Proposition 10).

Problem 14. Let (s_n) be a sequence in $\mathbb{R}$.
Show that $\lim |s_n| = 0$ if and only if $\lim s_n = 0$.

Problem 15. Let (s_n) and (t_n) be sequences in $\mathbb{R}$ such that $|s_n| \leq t_n$ for all n and $\lim t_n = 0$. Show that $\lim s_n = 0$.

Solution. Since $|s_n| \leq t_n$, we have $-t_n \leq s_n \leq t_n$.

Let $\epsilon > 0$ and let N be so large that $|t_n - 0| < \epsilon$ for $n > N$. Since

$$|t_n - 0| = |t_n| = |-t_n| = |-t_n - 0|,$$

then $|-t_n - 0| < \epsilon$ for $n > N$. Thus $\lim -t_n = 0$.

The result follows by the Squeeze Law. □

Problem 16. Let A be a bounded set of real numbers.

(a) Show that there exists a sequence in A which converges to $\sup A$.
(b) Show that there exists a sequence in A which converges to $\inf A$.

Problem 17. Let (a_n) and (b_n) be sequences in $\mathbb{R}$ such that (a_n) is bounded and $\lim b_n = 0$. Show that $\lim a_n b_n = 0$.

Solution. Let $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$. Since $\lim b_n = 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|b_n - 0| < \frac{\epsilon}{M}$. Then for $n > N$, we have

$$|a_n b_n - 0| = |a_n||b_n| \leq M\frac{\epsilon}{M} = \epsilon.$$

Thus $\lim a_n b_n = 0$. □

Problem 18. Construct sequences (a_n) and (b_n) of positive real numbers, with $c_n = a_n b_n$, satisfying

(0) $\lim_{n\to\infty} b_n = 0$;
(1) $\liminf c_n = 1$;
(2) $\limsup c_n = 2$.

Problem 19. Let (a_n) be a sequence of positive real numbers satisfying $a_{n+1}^2 = a_n$. Show that (a_n) converges to 1.

Definition 12. Let $A \subset \mathbb{R}$ be an open interval. A function $f : A \to \mathbb{R}$ is called a *contraction* if there exists $M \in \mathbb{R}$ such that $|f(a) - f(b)| \leq M|a - b|$ for any $a, b \in U$.

Problem 20. Let $f : \mathbb{R} \to \mathbb{R}$ be a contraction. Let (a_n) be a sequence of real numbers which converges to $p \in \mathbb{R}$. Show that $\lim f(a_n) = f(L)$.

Solution. Let $\epsilon > 0$. Since f is a contraction, there exists $M \in \mathbb{R}$ such that $|f(a) - f(b)| < M|a - b|$ for all $a, b \in \mathbb{R}$.

Since (a_n) converges to p, there exists $N \in \mathbb{N}$ such that $|a_n - p| < \frac{\epsilon}{M}$ for all $n > N$. Since f is a contraction,

$$|f(a_n) - f(p)| < M|a_n - p| < M\frac{\epsilon}{M} = \epsilon$$

for all $n > N$. Thus $f(a_n) \to f(p)$. □

Problem 21. Let (s_n) and (t_n) be sequences in $\mathbb{R}$.
Show that $\limsup(s_n + t_n) \le \limsup s_n + \limsup t_n$.

Solution. Let $S_m = \{s_n \mid n > m\}$, $T_m = \{t_n \mid n > m\}$, and $U_m = \{s_n + t_n \mid n > m\}$. We have $\sup(S_m + T_m) = \sup S_m + \sup T_m$ by Problem 4. But $U_m \subset S_m + T_m$, so $\sup U_m \le \sup S_m + \sup T_m$ by Problem 3. Thus

$$\begin{aligned}\limsup(s_n + t_n) &= \lim(\sup U_m) \\ &\le \lim(\sup S_m + \sup T_m) \\ &= \lim(\sup S_m) + \lim(\sup T_m) \\ &= \limsup s_n + \limsup t_n.\end{aligned}$$

□

Problem 22. Let (s_n) and (t_n) be bounded sequences over nonnegative real numbers.
Show that $\limsup s_n t_n \le (\limsup s_n)(\limsup t_n)$.

Solution. Let $S_m = \{s_n \mid n > m\}$, $T_m = \{t_n \mid n > m\}$, and $U_m = \{s_n t_n \mid n > m\}$. We have $\sup(S_m T_m) = (\sup S_m)(\sup T_m)$ by Problem 6. But $U_m \subset S_m T_m$, so $\sup U_m \le \sup S_m \sup T_m$ by Problem 3. Thus

$$\begin{aligned}\limsup(s_n t_n) &= \lim(\sup U_m) \\ &\le \lim(\sup S_m \sup T_m) \\ &= \lim(\sup S_m)\lim(\sup T_m) \\ &= \limsup(s_n)\limsup(t_n).\end{aligned}$$

□

Problem 23. Let $(s_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. Let $v = \liminf s_n$ and $u = \limsup s_n$. Show that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \ge N$, then $s_n \in (v - \epsilon, u + \epsilon)$.

Problem 24. Let (s_n) be a sequence of real numbers which converges to $s \in \mathbb{R}$. Let $\sigma_n = \frac{1}{n}\sum_{i=1}^{n} s_i$. Show that (σ_n) converges to s.

Solution. Let $\tau_n = \sigma_n - s$. It suffices to show that (τ_n) converges to zero. Note that

$$\tau_n = \frac{1}{n}\sum_{i=1}^{n} s_i - \frac{ns}{n} = \frac{1}{n}\sum_{i=1}^{n}(s_i - s).$$

Let $N_0 \in \mathbb{N}$ be so large that $|s_n - s| < \frac{\epsilon}{2}$ for all $n > N_0$. Let $M = \sum_{i=1}^{N} |s_i - s|$. Then for $n > N_0$, we have

$$\begin{aligned}|\tau_n| &\le \frac{M}{n} + \frac{1}{n}\sum_{i=N_0+1}^{n} |s_n - s| && \text{by } \Delta\text{-inequality} \\ &< \frac{M}{n} + \frac{1}{n}(n - N_0)\frac{\epsilon}{2} && \text{summing } n - N_0 \text{ small numbers} \\ &< \frac{M}{n} + \frac{\epsilon}{2} && \text{since } \frac{n - N_0}{n} \le 1.\end{aligned}$$

Now select $N \in \mathbb{N}$ with $N > N_0$ which is so large that $\frac{M}{n} < \frac{\epsilon}{2}$. Then for $n > N$, we have $|\tau_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This shows that $|\tau_n| \to 0$ as $n \to \infty$. Thus $\lim \tau_n = 0$. □

Problem 25. Let (a_n) and (b_n) be a sequences of real numbers we converge to a and b respectively. Let

$$\mu_n = \frac{a_1b_n + a_2b_{n-1} + \cdots + a_{n-1}b_2 + a_nb_1}{n}.$$

Show that (μ_n) converges to ab.

Solution. Let $\nu_n = \mu_n - ab$. It suffices to show that (ν_n) converges to zero.

Since (a_i) is a convergent sequence, is bounded; select $M > 0$ such that $|a_i| \leq M$. Also note that for any sequence (s_i), we have $\sum_{i=1}^n s_{n-i+1} = \sum_{i=1}^n s_i$; this follows from inductive use of commutativity.

Now

$$\begin{aligned}
|\nu_n| &= \frac{1}{n}|\sum_{i=1}^n a_ib_{n-i+1} - \frac{nab}{n}| \\
&= \frac{1}{n}|\sum_{i=1}^n (a_ib_{n-i+1} - ab)| \\
&\leq \frac{1}{n}\sum_{i=1}^n |a_ib_{n-i+1} - ab| \\
&= \frac{1}{n}\sum_{i=1}^n |a_ib_{n-i+1} - a_ib + a_ib - ab| \\
&\leq \frac{\sum_{i=1}^n |a_ib_{n-i+1} - a_ib|}{n} + \frac{\sum_{i=1}^n |a_ib - ab|}{n} \\
&\leq M\frac{\sum_{i=1}^n |b_{n-i+1} - b|}{n} + b\frac{\sum_{i=1}^n |a_i - a|}{n} \\
&= M\frac{\sum_{i=1}^n |b_i - b|}{n} + b\frac{\sum_{i=1}^n |a_i - a|}{n}.
\end{aligned}$$

Let $\tau_n = M\frac{\sum_{i=1}^n |b_i - b|}{n} + b\frac{\sum_{i=1}^n |a_i - a|}{n}$. By the Problem 24,

$$\lim_{n\to\infty} \tau_n = M\lim\frac{\sum_{i=1}^n |b_i - b|}{n} + b\lim\frac{\sum_{i=1}^n |a_i - a|}{n} = M \cdot 0 + b \cdot 0 = 0.$$

Since $0 \leq |\nu_n| \leq \tau_n$ and $\lim \tau_n = 0$, we have $|\nu_n| \to 0$ so $\lim \nu_n = 0$. $\square$

4. Cluster Points and Subsequences

4.1. Cluster Points.

Definition 13. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers and let $q \in \mathbb{R}$.
We say that $(a_n)_{n=1}^{\infty}$ *clusters at q* if

$$\forall \epsilon > 0 \; \forall N \in \mathbb{N} \; \exists n \geq N \; \ni \; |s_n - C| < \epsilon.$$

In this case, we call q a *cluster point* of $(a_n)_{n=1}^{\infty}$.

Proposition 19. *Let (a_n) be a sequence in $\mathbb{R}$ which converges to $p \in \mathbb{R}$. Then p is a cluster point of (a_n).*

Proof. Let $\epsilon > 0$ and $N \in \mathbb{N}$; we wish to show that there exists $n \geq N$ such that $|a_n - p| < \epsilon$. Since (a_n) converges to p, there exists $N_0 \in \mathbb{N}$ such that $n \geq N_0$ implies $|a_n - p| < \epsilon$. Let $n = \max\{N, N_0\}$; then $n \geq N$ and $|a_n - p| < \epsilon$. □

Proposition 20. *Let (a_n) be a bounded sequence of real numbers. Then*
- **(a)** $\limsup a_n$ *is a cluster point of* (a_n)*;*
- **(b)** $\liminf a_n$ *is a cluster point of* (a_n)*.*

Proof. Since (a_n) is bounded, $\limsup a_n$ and $\liminf a_n$ exist as real numbers. Let $u = \limsup a_n$; we wish to show that u is a cluster point of (a_n).

Let $\epsilon > 0$ and let $N \in \mathbb{N}$; it suffices to show that there exists $m \geq N$ such that $|a_m - u| < \epsilon$. Let $u_M = \sup\{a_n \mid n \geq M\}$.

Since $u = \lim_{M\to\infty} u_M$, there exists $N_0 \in \mathbb{N}$ such that, for all $M \geq N_0$, we have $|u_M - u| < \epsilon$. Let $M = \max\{N, N_0\}$. Then $U - \epsilon < u_M < u + \epsilon$; since $u_M = \sup\{a_n \mid n \geq M\}$, there exists an element of $\{a_n \mid n \geq M\}$ between $U - \epsilon$ and u_M. Select $m \in \mathbb{N}$ with $m \geq M \geq N$ such that $u - \epsilon < a_m < u_M$. We have $u - \epsilon < a_m < u + \epsilon$, so $|a_m - u| < \epsilon$. Thus u is a cluster point of (a_n).

That $\liminf a_n$ is a cluster point can be proved similarly. □

Proposition 21. (Bolzano-Weierstrass Theorem Version I)
Every bounded sequence of real numbers has a cluster point.

Proof. The limit superior of a bounded sequence exists as a real number, and this real number is a cluster point by Proposition 20. □

Proposition 22. *Let (a_n) be a bounded sequence in $\mathbb{R}$, and let q be a cluster point of (a_n). Then* $\liminf a_n \leq q \leq \limsup a_n$.

Proof. Suppose that $q \in \mathbb{R}$, and assume that $q > u = \limsup a_n$. Now $q - u$ is positive; let $\epsilon = \frac{q-u}{2}$. By definition of limit superior, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|\sup\{a_n \mid n \geq N\} - u| < \epsilon$. Thus for every $n \geq N$, we have $\sup\{a_n \mid n \geq N\} < u + \epsilon$, so $a_n < u + \epsilon = q - \epsilon$, and $q - a_n > \epsilon$.

This shows that q is not a cluster point; thus any cluster point must be less than or equal to $\limsup a_n$. Similarly, any cluster point must be greater than or equal to $\liminf a_n$. □

Proposition 23. *Let (a_n) be a sequence in $\mathbb{R}$. Then (a_n) converges to p if and only if p is the only cluster point of (a_n).*

Proof. We prove the double implication in each direction, using the fact that (a_n) converges (to p) if and only if $\liminf a_n = \limsup a_n$ (which equals p), as we have previously shown.

($\Rightarrow$) Suppose that (a_n) converges to p. By Proposition 19, p is a cluster point of (a_n), and we wish to show it is the only cluster point. Let q be a cluster point; we wish to show that $q = p$.

By Proposition 22, we have $\liminf a_n \leq q \leq \limsup a_n$. Because (a_n) converges to p, we know that $\liminf a_n = \limsup a_n = p$. Thus $q = p$.

($\Leftarrow$) Suppose that p is the only cluster point of (a_n). Then $\liminf a_n = p = \limsup a_n$. This shows that (a_n) converges to p. □

4.2. **Subsequences.** Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence of real numbers. A *subsequence* of a is the composition $a \circ n$ of a with a strictly increasing sequence $n : \mathbb{N} \to \mathbb{N}$ of positive integers.

If we denote the sequence a by $(a_n)_{n=1}^{\infty}$ and the sequence n by $(n_k)_{k=1}^{\infty}$, then we denote the subsequence by $(a_{n_k})_{k=1}^{\infty}$.

Proposition 24. *Let $n : \mathbb{N} \to \mathbb{N}$ such that $n \mapsto n_k$ be an increasing sequence. Then $n_k \geq k$.*

Proof. By induction on k.

For $k = 1$, we have $n_k = n_1 \geq 1$, since $n_k \in \mathbb{N}$.

Assume that $n_k \geq k$; then $n_k + 1 \geq k + 1$. Since n is increasing, $n_{k+1} > n_k$, so $n_{k+1} \geq n_k + 1$. Thus $n_{k+1} \geq n_k + 1 \geq k + 1$. □

Proposition 25. *Let (a_n) be a sequence of real numbers and let $p \in \mathbb{R}$. Then (a_n) converges to p if and only if every subsequence of (a_n) converges to p.*

Proof. We prove both directions.

($\Leftarrow$) Note that a sequence is a subsequence of itself. Thus if every subsequence of (a_n) converges to p, then in particular the sequence itself converges to p.

($\Rightarrow$) Suppose that $\lim a_n = p$. Let (a_{n_k}) be a subsequence of (a_n), and let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|a_n - p| < \epsilon$. Thus for $k \geq N$, we have $n_k \geq K \geq N$, so $|a_{n_k} - p| < \epsilon$. □

Proposition 26. *Let* (a_n) *be a sequence of real numbers. Then* (a_n) *has a monotonic subsequence.*

Proof. This proof follows Ross, which in turn follows D. J. Newman's *A Problem Seminar.*

Let's say that the i^{th} term of (a_n) is *dominant* if $a_j < a_i$ for every $j > i$.

Case 1: There are infinitely many dominant terms. In this case, set

$$n_1 = \min\{n \in \mathbb{N} \mid a_n \text{ is dominant}\}.$$

Then recursively set

$$n_{k+1} = \min\{n \in \mathbb{N} \mid a_n \text{ is dominant and } n > n_k\};$$

this set is nonempty by the hypothesis of this case. Then (a_{n_k}) is a decreasing sequence.

Case 2: There are finitely many dominant terms. In this case, set

$$n_0 = \max\{n \in \mathbb{N} \mid a_n \text{ is dominant}\}.$$

Then recursively set

$$n_{k+1} = \min\{n \in \mathbb{N} \mid a_n > a_{n_k} \text{ and } n > n_k\};$$

this set is nonempty because a_{n_0} was the last dominant term. Now (a_{n_k}) is an increasing sequence. □

Proposition 27. (Bolzano-Weierstrass Theorem Version II)
Every bounded sequence of real numbers has a convergent subsequence.

Proof. It is clear that if a sequence is bounded, then every subsequence is also bounded. Thus a bounded sequence has a bounded monotonic subsequence, which must converge. □

4.3. **Subsequential Limits.**

Definition 14. We say that q is a *subsequential limit* of (a_n) if there exists a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} a_{n_k} = q$.

Proposition 28. *Let* (a_n) *be a sequence of real numbers, and let* $q \in \mathbb{R}$. *Then* q *is a cluster point of* (a_n)*if and only if* q *is a subsequential limit of* (a_n).

Proof. Suppose that q is a cluster point. Then for every $N \in \mathbb{N}$ there exists $n \geq N$ such that $|a_n - q| < \frac{1}{N}$.

Set

$$n_1 = \min\{n \in \mathbb{N} \mid |a_n - q| < 1\},$$

and inductively set

$$n_{k+1} = \min\{n \in \mathbb{N} \mid |a_n - q| < \frac{1}{n} \text{ and } n > n_k\}.$$

That these sets are nonempty is assured by the fact that (a_n) clusters at q. Then (a_{n_k}) is a subsequence of (a_n) which converges to q.

Suppose that (a_{n_k}) is a subsequence which converges to q. Let $\epsilon > 0$ and let $N \in \mathbb{N}$. Let K be so large that $k \geq K \Rightarrow |a_{n_k} - q| < \epsilon$. Let $n = \max\{N, K\}$. Then $n \geq N$, so $n_k \geq N$. Moreover, $n \geq K$, so $n_k \geq K$ and $|a_{n_k} - q| < \epsilon$. □

Remark 3. We have previously seen that every bounded sequence has a cluster point, and we have just seen that every cluster point is the limit of a subsequence. This produces an alternate proof of the Bolzano-Weierstrass Theorem Version II.

Proposition 29. *Let (a_n) be a bounded sequence in $\mathbb{R}$.*
Then there exist monotonic subsequences of (a_n) which converge to $\limsup a_n$ *and* $\liminf a_n$.

Proof. We have seen that $\limsup a_n$ and $\liminf a_n$ are cluster points, and that cluster points are subsequential limits. Since every sequence has a monotonic subsequence, the result follows. ☐

4.4. **Problems.**

Problem 26. Construct a divergent sequence (a_n) of real numbers such that (a_{mk}) converges for every $m \in \mathbb{N}$, $m \geq 2$.

Solution. We use the fact that there are infinitely prime numbers.

Define

$$a_n = \begin{cases} 1 & \text{if } n \text{ is prime ;} \\ 0 & \text{otherwise .} \end{cases}$$

Since there are infinitely many primes, $\limsup a_n = 1$. Since there are infinitely many nonprimes, $\liminf a_n = 0$. Thus (a_n) does not converge.

However, for any $m \in \mathbb{N}$ with $m \geq 2$, mk is not prime for $k \geq 2$, so $a_{mk} = 0$ for all $k \geq 2$. Thus $\lim_{k\to\infty} a_{mk} = 0$, and (a_{mk}) converges. ☐

Problem 27. Let (a_n) and (b_n) be bounded sequences of positive real numbers, and suppose that $0 \in \mathbb{R}$ is a cluster point of the sequence $(a_n b_n)$. Show that 0 is a cluster point of either (a_n) or of (b_n).

Problem 28. Let (a_n) and (b_n) be bounded sequences of positive real numbers, and suppose that $c \in \mathbb{R}$ is a cluster point of the sequence $(a_n + b_n)$. Show that there exist cluster points a of (a_n) and b of (b_n) such that $c = a + b$.

Problem 29. Let (a_n) and (b_n) be bounded sequences of positive real numbers, and suppose that $c \in \mathbb{R}$ is a cluster point of the sequence $(a_n b_n)$. Show that there exist cluster points a of (a_n) and b of (b_n) such that $c = ab$.

Problem 30. Construct sequences (a_n) and (b_n) such that a is a cluster point of (a_n) and b is a cluster point of (b_n), but $a + b$ is not a cluster point of $(a_n + b_n)$.

Problem 31. Construct sequences (a_n) and (b_n) such that a is a cluster point of (a_n) and b is a cluster point of (b_n), but ab is not a cluster point of $(a_n b_n)$.

Problem 32. Let (a_n) and (b_n) be bounded sequences of positive real numbers, and suppose that $(a_n b_n)$ has a subsequence which converges to 0. Show that either (a_n) or (b_n) has a subsequence that converges to 0.

Problem 33. Let (a_n) and (b_n) be bounded sequences of positive real numbers, and suppose that $(a_n + b_n)$ has a subsequence which converges to $c \in \mathbb{R}$. Show that there exists a subsequence of (a_n) which converges to $a \in \mathbb{R}$ and a subsequence of (b_n) which converges to $b \in \mathbb{R}$ such that $c = a + b$.

Problem 34. Let (a_n) and (b_n) be bounded sequences of positive real numbers, and suppose that (a_nb_n) has a subsequence which converges to $c \in \mathbb{R}$. Show that there exists a subsequence of (a_n) which converges to $a \in \mathbb{R}$ and a subsequence of (b_n) which converges to $b \in \mathbb{R}$ such that $c = ab$.

5. Open and Closed Sets

5.1. Open Sets.

Definition 15. A subset $U \subset \mathbb{R}$ is called *open* if

$$\forall u \in U \; \exists \epsilon > 0 \; \ni \; |x - u| < \epsilon \Rightarrow x \in U.$$

This definition can be restated in terms of neighborhoods.

Definition 16. Let $x \in \mathbb{R}$. An *ϵ-neighborhood* of x is an open interval of the form $(x - \epsilon, x + \epsilon)$, where $\epsilon > 0$.

More generally, a *neighborhood* of x is a subset $Q \subset \mathbb{R}$ such that there exists $\epsilon > 0$ with $(x - \epsilon, x + \epsilon) \subset Q$.

So, a set $U \subset \mathbb{R}$ is open if every point in U is surrounded by an ϵ-neighborhood which is completely contained in U.

If $\mathcal{C}$ is a collection of subsets of a given set X, then the *union* and *intersubsection* of $\mathcal{C}$ are

$$\cup\mathcal{C} = \{x \in X \mid x \in C \text{ for some } C \in \mathcal{C}\};$$
$$\cap\mathcal{C} = \{x \in X \mid x \in C \text{ for all } C \in \mathcal{C}\}.$$

Proposition 30. *Let $\mathcal{T}$ denote the collection of all open subsets of $\mathbb{R}$. Then*

(a) $\varnothing \in \mathcal{T}$ *and* $\mathbb{R} \in \mathcal{T}$;
(b) *if* $\mathcal{O} \subset \mathcal{T}$, *then* $\cup\mathcal{O} \in \mathcal{T}$;
(c) *if* $\mathcal{O} \subset \mathcal{T}$ *is finite, then* $\cap\mathcal{O} \in \mathcal{T}$.

Proof.

(a) The condition for openness is vacuously satisfied by the empty set. For $\mathbb{R}$, consider $x \in \mathbb{R}$. Then $(x - 1, x + 1) \subset \mathbb{R}$. Thus $\mathbb{R}$ is open.

(b) Let $\mathcal{O} \subset \mathcal{T}$; that is, $\mathcal{O}$ is a collection of open sets. Select $x \in \cup\mathcal{O}$. Then $x \in U$ for some $U \in \mathcal{O}$. Since U is open, there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$. Since $U \subset \cup\mathcal{O}$, it follows that $(x - \epsilon, x + \epsilon) \subset \cup\mathcal{O}$. Thus $\cup\mathcal{O}$ is open.

(c) Let $\mathcal{O} \subset \mathcal{T}$ be a finite collection of open sets. Since $\mathcal{O}$ is finite, we may write $\mathcal{O} = \{U_1, U_2, \dots, U_n\}$, where U_i is an open set for $i = 1, \dots, n$. If $\cap\mathcal{O}$ is empty, we are done, so assume that it nonempty, and select $x \in \cap\mathcal{O}$. For each i, there exists ϵ_i such that $(x - \epsilon_i, x + \epsilon_i) \subset U_i$. Set $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\}$. Then $(x - \epsilon, x + \epsilon) \subset \cap\mathcal{O}$. Thus $\cap\mathcal{O}$ is open. $\square$

Proposition 31. *Let $\mathcal{O}$ be a collection of open intervals. If $\cap\mathcal{O}$ is nonempty, then $\cup\mathcal{O}$ is an open interval.*

Proof. By hypothesis, there exists $x \in \cap\mathcal{O}$. Write $\mathcal{O}$ as a family of sets:

$$\mathcal{O} = \{O_\alpha \mid \alpha \in A\},$$

where A is an indexing set. Now O_α is an open interval; we label its endpoints by letting $O_\alpha = (a_\alpha, b_\alpha)$, where $a_\alpha, b_\alpha \in \mathbb{R} \cup \{\pm\infty\}$. Set

$$a = \inf\{a_\alpha \mid \alpha \in A\} \quad \text{and} \quad b = \sup\{b_\alpha \mid \alpha \in A\}.$$

Claim: $\cup\mathcal{O} = (a, b)$. We prove both directions of containment.

($\subset$) Let $y \in \cup\mathcal{O}$. Then $y \in O_\alpha$ for some α. Thus $a \leq a_\alpha < y < b_\alpha \leq b$, so $y \in (a, b)$.

($\supset$) Let $y \in (a, b)$. Assume that $y \leq x$; the proof for $y \geq x$ is analogous. Now $a < y$, and since $a = \inf\{a_\alpha \mid \alpha \in A\}$, so there exists $\alpha \in A$ such that $a \leq a_\alpha < y$. Also $x \in O_\alpha$ so $a_\alpha < y \leq x < b_\alpha$; thus $y \in (a_\alpha, b_\alpha) = O_\alpha$, and $y \in \cup\mathcal{O}$. □

Proposition 32. *Let $U \subset \mathbb{R}$. Then U is open if and only if there exists a collection $\mathcal{O}$ of disjoint open intervals such that $U = \cup\mathcal{O}$.*

Proof. Let $a \in U$, and set $\mathcal{O}_a = \{O \subset U \mid O \text{ is an open interval and } a \in O\}$. Set $O_a = \cup\mathcal{O}_a$. By the previous proposition, O_a is an open interval.

Now suppose that $a, b \in U$ and suppose that $O_a \cap O_b \neq \varnothing$. Then there exists $c \in O_a \cap O_b$, so $O = O_a \cup O_b$ is an open interval by the Proposition 31. Also $a \in O$, so $O \in \mathcal{O}_a$, so $O \subset O_a$. Similarly, $O \subset O_b$. This shows that $O_a = O_b$.

Let $\mathcal{O} = \{O_a \mid a \in U\}$. This is a collection of disjoint open intervals contained in U, and every element of U is in one of these open intervals, so $U = \cup\mathcal{O}$. □

5.2. **Closed Sets.**

Definition 17. A subset $F \subset \mathbb{R}$ is *closed* if its complement $\mathbb{R} \setminus F$ is open.

We may characterize the collection $\mathcal{F}$ of closed subsets of $\mathbb{R}$ in a manner analogous to our characterization of $\mathcal{T}$, the collect of open subsets of $\mathbb{R}$, by the use of *DeMorgan's Laws.*

Proposition 33. (DeMorgan's Laws)
Let X be a set and let $\{A_\alpha \mid \alpha \in I\}$ be a family of subsets of X. Then

$$\bigcap_{\alpha\in I}(X \setminus A_\alpha) = X \setminus \Big(\bigcup_{\alpha\in I} A_\alpha\Big);$$

$$\bigcup_{\alpha\in I}(X \setminus A_\alpha) = X \setminus \Big(\bigcap_{\alpha\in I} A_\alpha\Big).$$

Proposition 34. *Let $\mathcal{F}$ denote the collection of all closed subsets of $\mathbb{R}$.*

- **(a)** $\varnothing \in \mathcal{F}$ *and* $\mathbb{R} \in \mathcal{F}$;
- **(b)** *if* $\mathcal{C} \subset \mathcal{F}$, *then* $\cap\mathcal{C} \in \mathcal{F}$;
- **(c)** *if* $\mathcal{C} \subset \mathcal{F}$ *is finite, then* $\cup\mathcal{C} \in \mathcal{T}$.

Proof. Apply DeMorgan's Laws to Proposition 30. □

Proposition 35. *Let $F \subset \mathbb{R}$. Then F is closed if and only if every sequence in F which converges in $\mathbb{R}$ has a limit in F.*

Proof. We prove both directions.

($\Rightarrow$) Suppose that F is closed, and let (a_n) be a sequence in F which converges to $a \in \mathbb{R}$. We wish to show that $p \in F$. Suppose not; then $p \in \mathbb{R} \smallsetminus F$. This set is open, so there exists $\epsilon > 0$ such that $(p - \epsilon, p + \epsilon) \subset \mathbb{R} \smallsetminus F$. Thus there exists $N \in \mathbb{N}$ such that $a_n \in \mathbb{R} \smallsetminus F$ for all $n \geq N$. This contradicts that the sequence is in F.

($\Leftarrow$) Suppose that F is not closed; we wish to construct a sequence in F which converges to a point not in F. Since F is not closed, then $\mathbb{R} \smallsetminus F$ is not open. This means that there exists a point $x \in \mathbb{R} \smallsetminus F$ such that for every $\epsilon > 0$, $(x - \epsilon, x + \epsilon)$ is not a subset of $\mathbb{R} \smallsetminus F$; that is, $(x - \epsilon, x + \epsilon)$ contains a point in F. For $n \in \mathbb{N}$, let $x_n \in (x - \frac{1}{n}, x + \frac{1}{n}) \cap F$. Then (x_n) is a sequence in F, but $\lim_{n \to \infty} x_n = x \notin F$. □

5.3. **Problems.**

Problem 35. Let (a_n) be a bounded sequence in $\mathbb{R}$ and let

$$\Lambda = \{q \in \mathbb{R} \mid q \text{ is a cluster point of } (a_n)\}.$$

Show that Λ is closed and bounded.

6. Continuity

6.1. **Continuity.**

Definition 18. Let $D \subset \mathbb{R}$. Let $f : D \to \mathbb{R}$ and $a \in D$. We say that f is *continuous at a* if

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \ni \; |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon.$$

Let $A \subset D$. We say that f is *continuous on A* if f is continuous at a for every $a \in A$, We say that f is *continuous* if f is continuous on its entire domain.

Observation 1. It is immediate that the condition for continuity can be rewritten as

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \ni \; f((a - \delta, a + \delta) \cap D) \subset (f(a) - \epsilon, f(a) + \epsilon),$$

where $f(U) = \{y \in \mathbb{R} \mid f(x) = y \text{ for some } x \in D\}$.

Proposition 36. *Let $D \subset \mathbb{R}$. Let $f : D \to \mathbb{R}$ and $a \in D$. Then f is continuous at a if and only if for every sequence $(x_n)_{n=1}^{\infty}$ in D which converges to a, the sequence $(f(x_n))_{n=1}^{\infty}$ converges to $f(a)$.*

Proof. We prove both directions.

($\Rightarrow$) Suppose that f is continuous at a, and let (x_n) be a sequence in D which converges to a. Let $\epsilon > 0$; we wish to find $N \in \mathbb{N}$ such that $n \geq N$ implies $|f(x_n) - f(a)| < \epsilon$.

Since f is continuous at a, there exists $\delta > 0$ such that for $x \in D$, $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. Let $N \in \mathbb{N}$ be so large that $n \geq N \Rightarrow |x_n - a| < \delta$. Then for $n \geq N$, we have $|f(x_n) - f(a)| < \epsilon$.

($\Leftarrow$) Suppose that f is not continuous at a. We wish to find a sequence (x_n) from D such that (x_n) converges to a, but $(f(x_n))$ does not converge to $f(a)$.

Since f is not continuous at a, there exists $\epsilon > 0$ such that for every $\delta > 0$ there exists $x \in (a - \delta, a + \delta)$ with $|f(x) - f(a)| \geq \epsilon$. Thus for each $n \in \mathbb{N}$, select $x_n \in (a - \frac{1}{n}, a + \frac{1}{n})$ such that $|f(x_n) - f(a)| \geq \epsilon$. Then (x_n) converges to a, but $(f(x_n))$ does not converge to $f(a)$. □

Proposition 37. *Let $D \subset \mathbb{R}$ be open and let $f : D \to \mathbb{R}$. Then f is continuous on D if and only if for every open set $V \subset \mathbb{R}$, the preimage $f^{-1}(V)$ is open.*

Proof. We prove both directions.

($\Rightarrow$) Suppose that f is continuous on D. Let $V \subset \mathbb{R}$ be open. We wish to show that the preimage

$$f^{-1}(V) = \{x \in D \mid f(x) \in V\}$$

is open. Let $a \in f^{-1}(V)$, so that $f(a) \in V$; we wish to find $\delta > 0$ such that $(a - \delta, a + \delta) \subset f^{-1}(V)$.

Since D is open, there exists $\delta_1 > 0$ such that $(a - \delta_1, a + \delta_1) \subset D$. Since V is open, there exists $\epsilon > 0$ such that $(f(a) - \epsilon, f(a) + \epsilon) \subset V$. Since f is continuous at a, there exists $\delta_2 > 0$ such that $|x - a| < \delta_2 \Rightarrow |f(x) - f(a)| < \epsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then for $x \in (a - \delta, a + \delta)$, we have $x \in D$, and $|x - a| < \delta$, so $|f(x) - f(a)| < \epsilon$, so $f(x) \in V$. Thus $x \in f^{-1}(V)$.

($\Leftarrow$) Suppose that for every open set $V \subset \mathbb{R}$, the preimage $f^{-1}(V)$ is open. Let $a \in D$; we wish to show that f is continuous at a. Let $\epsilon > 0$; we wish to find $\delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$.

Let $V = (f(a) - \epsilon, f(a) + \epsilon)$. Now $f^{-1}(V)$ is open, and $a \in f^{-1}(V)$, so there exists $\delta > 0$ such that $(a - \delta, a + \delta) \subset f^{-1}(V)$. Let $x \in D$ such that $|x - a| < \delta$. Then $x \in (a - \delta, a + \delta)$, so $x \in f^{-1}(V)$, so $f(x) \in V$, which says that $|f(x) - f(a)| < \epsilon$. □

Proposition 38. *Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$, and $k \in \mathbb{R}$. If f and g are continuous at a, then kf, $f + g$ and fg are continuous at a. If additionally $g(a) \neq 0$, then $\frac{f}{g}$ is continuous at a.*

Proof. Assume that f and g are continuous at a; we prove that $f + g$ is continuous at a, the other proofs being almost identical in concept.

Let (x_n) be a sequence from D which converges to a. It suffices to show that $((f + g(x_n))$ converges to $(f + g)(a)$.

Then $(f(x_n))$ converges to $f(a)$, and $(g(x_n))$ converges to $g(a)$. By the sum of limits theorem for sequences,

$$\lim((f+g)(x_n)) = \lim(f(x_n)+g(x_n)) = \lim f(x_n)+\lim g(x_n) = f(a)+g(a) = (f+g)(a).$$

Thus $f + g$ is continuous at a. □

Remark 4. This proposition, in concert with liberal use of induction, can be used to show that a polynomial function is continuous on $\mathbb{R}$, and that a rational function is continuous on its domain.

6.2. **Connectedness.**

Definition 19. Let $A \subset \mathbb{R}$. We say that A is *connected* if for all $x_1, x_2 \in I$ with $x_1 < x_2$, we have $[x_1, x_2] \subset A$.

Proposition 39. *Let $a, b \in \mathbb{R}$ with $a < b$. Then $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ is connected.*

Proof. Let $x_1, x_2 \in [a, b]$ with $x_1 < x_2$. Then $a \leq x_1 < x_2 \leq b$.

Let $x \in [x_1, x_2]$. By transitivity of order, $a \leq x \leq b$. Thus $x \in [a, b]$. This shows that $[x_1, x_2] \subset [a, b]$. Thus $[a, b]$ is connected. □

Observation 2. Actually, one can extend this prove into cases to see that a subset of $\mathbb{R}$ is connected if and only if it is an interval.

Proposition 40. *Let $f : D \to \mathbb{R}$ be continuous, and let $I \subset D$ be connected. Then $f(I)$ is connected.*

Proof. Let $y_1, y_2 \in f(I)$ with $y_1 < y_2$, and let $y \in [y_1, y_2]$. We wish to show that $y \in f(I)$.

Let $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$. We assume that $x_1 < x_2$, the other case being similar. Since I is connected, $[x_1, x_2] \subset I$.

Let $S = \{x \in [x_1, x_2] \mid f(x) \leq y\}$. Now $f(x_1) = y_1 \leq y$, so $x_1 \in S$, and S is nonempty. Also x_2 is an upper bounded for S.

Let $s = \sup S$; then $x_1 \leq s \leq x_2$, so $s \in I$. For every $n \in \mathbb{N}$, there exists $s_n \in S$ such that $s - \frac{1}{n} < s_n \leq s$. The sequence (s_n) clearly converges to s, and since f is continuous, the sequence $(f(s_n))$ converges to $f(s)$. Since $f(s_n) \leq y$ for all n, $f(s) \leq y$.

If $s = x_2$, then $f(s) = y_2 \geq y \geq f(s)$, so $y = y_2 \in f(I)$. Thus assume that $s < x_2$, and let $d = x_2 - s$. Set $t_n = s + \frac{d}{n}$. Then $t_n \in [x_1, x_2]$ but $t_n > s$, so $t_n \notin S$, and $f(t_n) > y$. Moreover (t_n) clearly converges to s, and since f is continuous, $(f(t_n))$ converges to $f(s)$. Thus $f(s) \geq y$. Ultimately, this shows that $y = f(s)$, so $y \in f(I)$. □

Theorem 3. (Intermediate Value Theorem)
Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a)f(b) < 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$.

Proof. Since $[a, b]$ is connected, so is $f([a, b])$

Since $f(a)f(b) < 0$, either $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$. If $f(a) < 0 < f(b)$, then $0 \in [f(a), f(b)] \subset f([a, b])$, and if $f(b) < 0 < f(a)$, then $0 \in [f(a), f(b)] \subset f([a, b])$. In either case, $0 \in f([a, b])$, so there exists $c \in E$ such that $f(c) = 0$. Clearly $c \neq a$ and $c \neq b$, so $c \in [a, b] \setminus \{a, b\} = (a, b)$. □

6.3. **Compactness.**

Definition 20. Let $A \subset \mathbb{R}$. We say that A is *compact* if it is closed and bounded.

Proposition 41. *Let $K \subset \mathbb{R}$ be compact. Then* $\inf K, \sup K \in K$.

Proof. First note that $\inf K$ and $\sup K$ exist as real number, since K is bounded. We have seen that there exist sequences in K which converge to $\inf K$ and $\sup K$, respectively. Since K is closed, the limits of these sequences are in K. □

Proposition 42. *Let $f : D \to \mathbb{R}$ be continuous, and let $K \subset D$ be compact. Then $f(K)$ is bounded.*

Proof. Suppose that $f(K)$ is unbounded, and assume that $f(K)$ is unbounded above; the argument if $f(K)$ is unbounded below is analogous.

For every $n \in \mathbb{N}$, there exists $y_n \in f(K)$ such that $y_n \geq n$. Clearly (y_n) is unbounded, and every subsequence of (y_n) is unbounded.

For every $n \in \mathbb{N}$, select $x_n \in f^{-1}(y_n)$. Then (x_n) is a sequence in K. Since K is bounded, (x_n) has a convergent subsequence, say (x_{n_k}) converges to x. Since K is closed, $x \in K$; thus f is defined at x, and $f(x) \in f(K)$. Let $y = f(x)$. Since f is continuous at x, $(f(x_{n_k}))$ converges to y. Since $y_{n_k} = f(x_{n_k})$, the subsequence (y_{n_k}) converges to y. But then (y_{n_k}) is bounded, a contradiction. Therefore, $f(K)$ must be bounded. □

Proposition 43. *Let $f : D \to \mathbb{R}$ be continuous, and let $K \subset D$ be compact. Then $f(K)$ is compact.*

Proof. We just showed that $f(K)$ is bounded; now we show that $f(K)$ is closed.

Let (y_n) be a sequence in $f(K)$ which converges to p; we show that $p \in K$. Select $x_n \in f^{-1}(y_n)$. Then (x_n) is a sequence in K. Since K is bounded, this sequence is bounded, so it has a convergent subsequence, say (x_{n_k}) converges to q. Since K is closed, $q \in K$. Now f is continuous as q, so $(f(x_{n_k})) = (y_{n_k})$ converges to $f(q)$. Since (y_{n_k}) is a subsequence of (y_n) and (y_n) converges to p, the subsequence (y_{n_k}) must converge to p. This shows that $f(q) = p$. Since $q \in K$, $p = f(q) \in f(K)$. □

Proposition 44. *Let $f : D \to \mathbb{R}$ be continuous, and let $K \subset D$ be compact. Then there exist $x_1, x_2 \in K$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for every $x \in K$.*

Proof. Note that $f(K)$ is compact, so it is closed and bounded. Let $y_1 = \inf f(K)$ and $y_2 = \sup f(K)$, which exist as real numbers because $f(K)$ is bounded. Since $f(K)$ is compact, $y_1, y_2 \in f(K)$. Let $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$, and conclude that for $x \in K$, we have $f(x) \in f(K)$, so

$$f(x_1) = y_1 = \inf f(K) \leq f(x) \leq \sup f(K) = y_2 = f(x_2).$$

□

Remark 5. Our last three principal results can be paraphrased as follows:

- **(a)** the continuous image of a connected set is connected;
- **(b)** the continuous image of a compact set is compact;
- **(c)** a continuous function on a compact set attains a minimum and maximum on that set.

6.4. **Continuity Examples.**

Example 1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that f is discontinuous at every real number.

Proof. Let $x_0 \in \mathbb{R}$. To show that f is discontinuous at x_0, it suffices to find $\epsilon > 0$ such that for every $\delta > 0$, there exists $x \in (x_0 - \delta, x_0 + \delta)$ with $|f(x) - f(x_0)| \geq \epsilon$.

Let $\epsilon = \frac{1}{2}$ and let $\delta > 0$. Then there exists both a rational and an irrational in $(x_0 - \delta, x_0 + \delta)$. If x_0 is rational, let x_1 be an irrational in this interval, and we have $|f(x_1) - f(x_0)| = 1 > \epsilon$; if x_0 is irrational, let x_2 be a rational in this interval, and we still have $|f(x_2) - f(x_0)| = 1 > \epsilon$. Thus f is not continuous at x_0. □

Example 2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that f is continuous at $x = 0$ and discontinuous at all nonzero real numbers.

Proof. Let $x_0 \in \mathbb{R} \setminus \{0\}$; we show that f is discontinuous at x_0. Let $\epsilon = \frac{|x_0|}{2}$ and let $\delta > 0$. Then there exists both a rational and an irrational in $(x_0 - \delta, x_0 + \delta)$. If x_0 is rational, let x_1 be an irrational in this interval, and we have $|f(x_1) - f(x_0)| = |x_0| > \epsilon$. If x_0 is irrational, let x_2 be a rational in this interval such that $|x_2| > |x_0|$ and we still have $|f(x_2) - f(x_0)| = |x_2| > |x_0| > \epsilon$. Thus f is not continuous at x_0.

Now we consider the behavior of f at zero. Let $\epsilon > 0$ and let $\delta = \epsilon$. Then if $|x - 0| < \delta$, we have $|f(x) - f(0)| = 0$ if x is irrational and $|f(x) - f(0)| = |x|$ if x is rational; in either case, $|f(x) - f(0)| \leq |x| < \delta = \epsilon$, so f is continuous at zero. □

Example 3. If $r \in \mathbb{Q}$, there exists $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $r = \frac{p}{q}$. Define $q : \mathbb{Q} \to \mathbb{R}$ by

$$q(r) = \min\{q \in \mathbb{N} \mid r = \frac{p}{q} \text{ for some } p \in \mathbb{Z}\}.$$

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q(x)} & \text{if } x \text{ is rational} \end{cases}$$

Show that f is discontinuous at every rational and continuous at every irrational.

Proof. Suppose that x_0 is rational. We wish to show that f is not continuous at x_0. It suffices to find $\epsilon > 0$ such that for every $\delta > 0$ there exists $x_1 \in (x_0 - \delta, x_0 + \delta)$ with $|x_0 - x_1| > \epsilon$.

Since x_0 is rational, we have $x_0 = \frac{p}{q(x_0)}$ for some $p \in \mathbb{Z}$. Let $\epsilon = \frac{1}{2q(x_0)}$ and let $\delta > 0$. Then $(x_0 - \delta, x_0 + \delta)$ contains an irrational number, say x_1; then $|x_0 - x_1| < \delta$ but $|f(x_0) - f(x_1)| = \frac{1}{q(r)} > \epsilon$. Thus f cannot be continuous at x_0.

Suppose that x_0 is irrational. Let $\epsilon > 0$. It suffices to find $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$.

Let $N \in \mathbb{N}$ be so large that $\frac{1}{N} < \epsilon$. Let a be the greatest integer which is less than x_0 and b be the least integer which is greater than x_0; then $b = a + 1$ and $x_0 \in [a, b]$.

For $q \in \mathbb{Q}$, there exist only finitely many points in the set $[a, b] \cap \{\frac{k}{q} \mid k \in \mathbb{Z}\}$ (in fact, this set contains no more than q points). Thus the set

$$D = [a, b] \cap \{\frac{k}{q} \mid k \in \mathbb{Z}, q \leq N\}$$

is finite (there are no more than $\frac{N(N+1)}{2}$ points in this set). Let

$$\delta = \min\{|x_0 - d| \mid d \in D\};$$

since this set is a finite set of positive real numbers, the minimum exists as a positive real number. Then $(x_0 - \delta, x_0 + \delta) \subset [a, b]$. Let $x \in (x_0 - \delta, x_0 + \delta)$. If x is irrational, we have $|f(x) - f(x_0)| = 0 < \epsilon$, and if x is rational, we have $|f(x) - f(x_0)| = \frac{1}{q(x)} < \frac{1}{N} < \epsilon$. Thus f is continuous at x_0. □

6.5. **Problems.**

Problem 36. Let $f : [0, \infty) \to \mathbb{R}$ be given by $f(x) = \sqrt{x}$, and let $a \in \mathbb{R}$ be positive. Show that f is continuous at a. (hint: use Proposition 36).

Problem 37. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) = x^2$ for every $x \in \mathbb{Q}$. Show that $f(x) = x^2$ for every $x \in \mathbb{R}$ (hint: use Proposition 36).

Problem 38. Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous. Suppose that $f(a) = g(b)$ and $f(b) = g(a)$. Show that there exists $c \in (a, b)$ such that $f(c) = g(c)$ (hint: use Theorem 3).

Problem 39. Let $K \subset \mathbb{R}$ be a compact connected set. Show that there exist $a, b \in \mathbb{R}$ with $a \leq b$ such that $K = [a, b]$.

Definition 21. Let $D \subset \mathbb{R}$ and let $a \in D$. Let $f : D \to \mathbb{R}$. We say that a is a *fixed point* of f if $f(a) = a$.

Problem 40. Let K be a compact connected subset of $\mathbb{R}$, and let $f : K \to \mathbb{R}$ be continuous. Show that f has a fixed point (hint: use Theorem 3).

Department of Mathematics and CSci, Southern Arkansas University
E-mail address: `plbailey@saumag.edu`